Characteristic function noncentral chi-square distribution:

Let us assume that we have the following function where X is normally distributed:

$$X \sim N(\mu, \sigma^2)$$
$$Y = X^2$$

We can state the following:

$$F_y(y) = P(Y \leq y) = P(X^2 \leq y) = P\left(|X| \leq \sqrt{y}\right) = F_x\left(\sqrt{y}\right) - F_x(-\sqrt{y})$$

By taking the derivative of the cumulative distribution function of $Y$, we get:

$$\frac{d}{dy} F_y(y) = \frac{d}{dy}\left(F_x\left(\sqrt{y}\right) - F_x\left(-\sqrt{y}\right)\right) = \frac{1}{2\sqrt{y}} P_x\left(\sqrt{y}\right) + \frac{1}{2\sqrt{y}} P_x(-\sqrt{y})$$

As the PDF of the stochastic variable $X$ is even (It is symmetric due to the fact that it has a normal distribution), we can state:

$$= \frac{P_x(\sqrt{y})}{\sqrt{y}}$$

This results in:

$$\frac{P_x(\sqrt{y})}{\sqrt{y}} = \frac{1}{\sigma\sqrt{2\pi y}} e^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} \quad with\, y \geq 0$$

We now can find the characteristic function through the moment generating function:

$$M_y(it) = E\left[e^{itY}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_0^\infty \frac{1}{\sqrt{y}} e^{-\frac{(\sqrt{y}-\mu)^2}{2\sigma^2}} e^{ity} dy$$

We now make the following convenient substitutions:

$$a = \frac{1}{\sigma\sqrt{2\pi}}$$
$$c = \sqrt{2}\sigma$$
$$= a \int_0^\infty \frac{1}{\sqrt{y}} e^{-\frac{(\sqrt{y}-\mu)^2}{c^2}} e^{ity} dy$$

Using the substitution rule:

$$\alpha = \sqrt{y} - \mu$$
$$\frac{d\alpha}{dy} = \frac{1}{2\sqrt{y}}$$
$$2\sqrt{y}\, d\alpha = dy$$
$$y = (\alpha + \mu)^2$$

This results in:

$$= 2a \int_0^\infty e^{-\frac{\alpha^2}{c^2}} e^{it(\alpha+\mu)^2} d\alpha = 2a \int_0^\infty e^{-\frac{\alpha^2}{c^2}} e^{it\alpha^2} e^{it2\alpha\mu} e^{it\mu^2} d\alpha = 2ae^{it\mu^2} \int_0^\infty e^{-\frac{\alpha^2}{c^2}} e^{it\alpha^2} e^{it2\alpha\mu} d\alpha$$

$$= 2ae^{it\mu^2} \int_0^\infty e^{-\frac{1}{C^2}(\alpha^2 - itc^2\alpha^2 - it2c^2\alpha\mu)} d\alpha = 2ae^{it\mu^2} \int_0^\infty e^{-\frac{1}{C^2}(\alpha^2(1-itc^2) - it2c^2\alpha\mu)} d\alpha$$

We now make the convenient substitution:

$$\tau = 1 - itc^2$$

$$= 2ae^{it\mu^2} \int_0^\infty e^{-\frac{1}{C^2}(\alpha^2\tau - it2c^2\alpha\mu)} d\alpha = 2ae^{it\mu^2} \int_0^\infty e^{-\frac{\tau}{C^2}(\alpha^2 - 2\alpha\frac{itc^2\mu}{\tau})} d\alpha$$

We can conveniently rewrite the integral to:

$$= 2ae^{it\mu^2} \int_0^\infty e^{-\frac{\tau}{C^2}(\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2 + \frac{t^2c^4\mu^2}{\tau^2})} d\alpha$$

$$= 2ae^{it\mu^2} e^{-\frac{t^2c^2\mu^2}{\tau}} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha$$

$$= 2ae^{t\mu^2(i - \frac{tc^2}{\tau})} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha$$

$$= 2ae^{t\mu^2(\frac{i\tau - tc^2}{\tau})} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha$$

Now we substitute $\tau$ in the numerator of the first exponential:

$$= 2ae^{t\mu^2(\frac{i(1-itc^2) - tc^2}{\tau})} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha$$

$$= 2ae^{t\mu^2(\frac{i + tc^2 - tc^2}{\tau})} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha = 2ae^{\frac{it\mu^2}{\tau}} \int_0^\infty e^{-\frac{\tau}{C^2}\left(\alpha - \frac{itc^2\mu}{\tau}\right)^2} d\alpha$$

We use the substitution rule:

$$\varphi = \alpha - \frac{itc^2\mu}{\tau}$$
$$\frac{d\varphi}{d\alpha} = 1$$

$$= 2ae^{\frac{it\mu^2}{\tau}} \int_0^\infty e^{-\frac{\tau}{C^2}\varphi^2} d\varphi$$

Making a convenient substitution:

$$\beta = \frac{C}{\sqrt{\tau}}$$
$$= 2ae^{\frac{it\mu^2}{\tau}} \int_0^\infty e^{-\frac{\varphi^2}{\beta^2}} d\varphi$$

Using substitution rule:

$$\gamma = \frac{\varphi}{\beta}$$
$$\frac{d\gamma}{d\varphi} = \frac{1}{\beta}$$
$$\beta d\gamma = d\varphi$$
$$= 2a\beta e^{\frac{it\mu^2}{\tau}} \int_0^\infty e^{-\gamma^2} d\gamma$$

We can clearly see that the integrand is symmetric. By extending the upper and lower limit of this Riemann integral to $\infty$ and $-\infty$ and taking the $\frac{1}{2}$ of the result we can find the answer. This extension leads to the well known Gaussian integral. For more information I would like to redirect to my website http://www.planetmathematics.com where one can find a document about this integral.

$$= 2a\beta e^{\frac{it\mu^2}{\tau}} \frac{1}{2} \int_{-\infty}^{\infty} e^{-\gamma^2} d\gamma = a\beta e^{\frac{it\mu^2}{\tau}} \sqrt{\pi}$$

Resubstituting all substitutions gives:

$$= \frac{1}{\sigma\sqrt{2\pi}} \frac{\sqrt{2}\sigma}{(1 - i2\sigma^2 t)^{\frac{1}{2}}} e^{\frac{it\mu^2}{(1-i2\sigma^2 t)}} \sqrt{\pi} = \frac{1}{(1 - i2\sigma^2 t)^{\frac{1}{2}}} e^{\frac{it\mu^2}{(1-i2\sigma^2 t)}}$$

This is the characteristic function of the non-central chi-square distribution with one degree of freedom.

Normally when we see the definition of the chi-square distribution we have the following:

$$Y = X_1^2 + X_2^2 + \cdots + X_n^2$$

Where $X_i$ is independent, with the distribution:

$$X_i \sim N(\mu_i, \sigma^2)$$

One advantage of Fourier transforming (calculating the characteristic function) of a summation of independent stochastic variables is that their respective characteristic functions can be multiplied in the Fourier Domain (or convolution in the spatial domain).

Proof:

$$M_y(it) = E\left[e^{itY}\right] = E\left[e^{it\sum_{i=1}^{n} X_i^2}\right] = E\left[e^{itX_1^2} \cdots e^{itX_n^2}\right]$$

Due to the independence of the stochastic variables and the properties of the expectance operator:

$$E\left[e^{itX_1^2} \cdots e^{itX_n^2}\right] = E\left[e^{itX_1^2}\right] \cdots E\left[e^{itX_n^2}\right] = \frac{1}{(1 - i2\sigma^2 t)^{\frac{N}{2}}} e^{\frac{it\sum_{i=1}^{n}\mu_i^2}{(1-i2\sigma^2 t)}}$$

Hence the characteristic function of the noncentral chi-square distribution.