Characteristic function normal distribution

We first start with the moment generating function or characteristic function:

$$M_x(it) = E\left[e^{itx}\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{2\sigma^2}} e^{itx}\, dx$$

We now make the simple substitution:

$$a = \frac{1}{\sigma\sqrt{2\pi}}$$
$$c = \sqrt{2}\sigma$$

This results in:

$$= a \int_{-\infty}^{\infty} e^{-\frac{(x-\mu)^2}{c^2}} e^{itx}\, dx$$

Using the substitution rule:

$$\alpha = x - \mu$$
$$\frac{d\alpha}{dx} = 1$$

Resulting in:

$$= a \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{c^2}} e^{it(\alpha+\mu)}\, d\alpha = ae^{it\mu} \int_{-\infty}^{\infty} e^{-\frac{\alpha^2}{c^2}} e^{it\alpha}\, d\alpha$$
$$= ae^{it\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{c^2}(\alpha^2 - itc^2\alpha)}\, d\alpha$$

Resubstituting c between the brackets:

$$= ae^{it\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{c^2}(\alpha^2 - i2t\sigma^2\alpha)}\, d\alpha = ae^{\left(it\mu - \frac{t^2\sigma^2}{2}\right)} \int_{-\infty}^{\infty} e^{-\frac{(\alpha - it\sigma^2)^2}{c^2}}\, d\alpha$$

Using the substitution rule:

$$\gamma = \alpha - it\sigma^2$$
$$\frac{d\gamma}{d\alpha} = 1$$
$$= ae^{(it\mu - \frac{t^2\sigma^2}{2})} \int_{-\infty}^{\infty} e^{-\frac{\gamma^2}{c^2}}\, d\gamma$$

Using the substitution rule again (for the sake of simplicity I did not combine this step and the step above):

$$\varphi = \frac{\gamma}{c}$$
$$\frac{d\varphi}{d\gamma} = \frac{1}{c}$$
$$c\, d\varphi = d\gamma$$
$$= ace^{(it\mu - \frac{t^2\sigma^2}{2})} \int_{-\infty}^{\infty} e^{-\varphi^2}\, d\gamma$$

We can easily see that this integral is equal to the Gaussian integral. If this is not familiar to you, please view the document about the Gaussian integral on my website (http://www.planetmathematics.com).

$$= ace^{(it\mu - \frac{t^2\sigma^2}{2})}\sqrt{\pi}$$

Now we resubstitute a and c and obtain the characteristic function of the univariate normal distribution.

$$e^{(it\mu - \frac{t^2\sigma^2}{2})}\ \frac{1}{\sigma\sqrt{2\pi}}\ \sqrt{2}\sigma\ \sqrt{\pi} = e^{(it\mu - \frac{t^2\sigma^2}{2})}$$